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Mittag-Leffler coherent states

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Abstract. We create a family of boson coherent states using the functions of Mittag-Leffler (ML) $E_\alpha(z)$, ($\alpha > 0$) and their generalizations $E_{\alpha,\beta}(z)$, ($\alpha, \beta > 0$) instead of exponentials. These states are shown to satisfy the usual requirements of normalizability, continuity in the label and the resolution of unity with a positive weight function. This last quantity is found for arbitrary $\alpha, \beta > 0$ as a solution of an associated Stieltjes moment problem. In addition, for $\alpha = m = 1, 2, 3\ldots$ and $\beta = 1$ (corresponding to $E_m(z)$) we propose and analyse special q-deformations ($0 < q \leqslant 1$) of the functions $E_m(z)$ which serve as a tool to define q-deformed coherent states of ML type. We provide the expressions for expectation values of physical quantities for all the above states. We discuss physical properties of these states, noting that they are squeezed. The ML coherent states are sub-Poissonian in nature, whereas the q-deformed ML states can be sub- and super-Poissonian depending on q. All these states are shown to be eigenstates of deformed boson operators whose commutation relations are given.

1. Introduction

Conventional coherent states constitute a family of collective states of the harmonic oscillator parametrized by a single complex number z. The huge literature on this subject can be traced back from [1, 2]. The normalized coherent states, labelled by the complex number z, are defined by

$$|z\rangle = \mathcal{N}^{-\frac{1}{2}}(|z|^2)\exp(z\hat{a}^\dagger)|0\rangle \tag{1}$$

$$= \mathrm{e}^{-\frac{|z|^2}{2}}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n\rangle \tag{2}$$

where $\hat{N} = \hat{a}^\dagger\hat{a}$, $[\hat{a}, \hat{a}^\dagger] = 1$, $\hat{N}|n\rangle = n|n\rangle$ and $\langle n|n'\rangle = \delta_{n,n'}$; $|0\rangle$ is the ground state of the harmonic oscillator.

Their normalization is $\mathcal{N}(|z|^2) = \exp(|z|^2)$ and the states have non-vanishing overlap,

$$\langle z|z'\rangle = \exp(-\tfrac{1}{2}|z - z'|^2 + \mathrm{i}\,\mathrm{Im}\,(z^*z')). \tag{3}$$

The states $|z\rangle$ are eigenstates of a non-Hermitian operator $\hat{a}$, $\hat{a}|z\rangle = z|z\rangle$. The resolution of unity in terms of orthogonal projectors $|n\rangle\langle n|$

$$\sum_{n=0}^{\infty}|n\rangle\langle n| = I \tag{4}$$

has to be replaced by a *weighted* sum of non-orthogonal projectors $|z\rangle\langle z|$, i.e. by

$$\iint_{\mathbb{C}} \mathrm{d}^2 z\, |z\rangle \tilde{W}(|z|^2)\langle z| = I \tag{5}$$

with $\mathrm{d}^2 z = \mathrm{d}(\mathrm{Re}\, z)\, \mathrm{d}(\mathrm{Im}\, z)$, where $\tilde{W}(|z|^2)$ is a positive function to be determined from (2) and (5). The infinite set of conditions (5) can be rewritten as

$$\sum_{n=0}^{\infty} \left[\frac{\pi}{n!} \int_0^{\infty} \mathrm{e}^{-x} x^n \tilde{W}(x)\, \mathrm{d}x \right] |n\rangle\langle n| = I \tag{6}$$

with $x \equiv |z|^2$, which boils down to a (trivial) Stieltjes power-moment problem [3]: determine a positive $W(x) = \pi\, \mathrm{e}^{-x} \tilde{W}(x)$ such that

$$\int_0^{\infty} x^n W(x)\, \mathrm{d}x = n! \qquad n = 0, 1, 2 \ldots \infty \tag{7}$$

which yields a (unique) solution $W(x) = \mathrm{e}^{-x}$. Note that with equation (2), (7) reads

$$\int_0^{\infty} x^n \left[\pi \frac{\tilde{W}(x)}{\mathcal{N}(x)} \right] \mathrm{d}x = n! \qquad n = 0, 1, 2 \ldots \infty. \tag{8}$$

A quite general definition of coherent states has been given by Klauder [4], who proposed the following minimal set of conditions.

The states $|\lambda\rangle$ are coherent states if the following are satisfied:

(a) $|\lambda\rangle$ are normalizable, i.e. $\langle\lambda|\lambda\rangle = 1$;
(b) $|\lambda\rangle$ are continuous in the label λ, i.e. $|\lambda - \lambda'| \to 0 \Longrightarrow \| |\lambda\rangle - |\lambda'\rangle \|^2 \to 0$;
(c) the set $|\lambda\rangle$ allows a resolution of unity with a positive weight function $\tilde{W}(|\lambda|^2) > 0$, such that

$$\iint_{\mathbb{C}} d^2\lambda |\lambda\rangle \tilde{W}(|\lambda|^2)\langle\lambda| = I. \tag{9}$$

Condition (a) is axiomatic for allowable vectors, and condition (b) follows from the continuity of the overlapping factor $\langle\lambda|\lambda'\rangle$ through

$$\| |\lambda\rangle - |\lambda'\rangle \|^2 = 2(1 - \mathrm{Re}\, \langle\lambda|\lambda'\rangle) \tag{10}$$

and is easily satisfied in practice. Condition (c) imposes very severe restrictions on possible sets $|\lambda\rangle$. In particular, if

$$|\lambda\rangle = \mathcal{N}^{-\frac{1}{2}}(|\lambda|^2) \sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{\rho(n)}} |\lambda\rangle \qquad \rho(0) = 1 \tag{11}$$

then

$$\mathcal{N}(|\lambda|^2) = \sum_{n=0}^{\infty} \frac{|\lambda|^{2n}}{\rho(n)} > 0 \tag{12}$$

and (c) imposes for $x \equiv |\lambda|^2$

$$\int_0^{\infty} x^n \left[\pi \frac{\tilde{W}(x)}{\mathcal{N}(x)} \right] \mathrm{d}x = \rho(n) \qquad n = 0, 1, 2 \ldots \infty \tag{13}$$

which is the Stieltjes power-moment problem for $W(x) \equiv \pi \tilde{W}(x)/\mathcal{N}(x)$, $W(x) > 0$, [3]. An inverse approach is also possible: any positive function $V(x)$ possessing moments such that the normalization equation (12) converges, can serve as a weight function for some class of coherent states, often with very interesting geometric properties [4, 5]. A yet more ambitious method is to choose some moment set and attempt to solve it for the weight function. This method was adopted in [6], where a special form of moments $\rho(n; q)$ with a parameter $0 < q \leqslant 1$ was shown to lead to a soluble moment problem.

In this work we extend the states (2) in the sense of replacing $n!$ by its natural generalization $\Gamma(\alpha n+\beta)$, $(\alpha, \beta > 0)$; the normalization $\exp(|z|^2)$ becomes an entire function $\Gamma(\beta)E_{\alpha,\beta}(|z|^2)$ where $E_{\alpha,\beta}(|z|^2)$ is a generalized Mittag-Leffler (ML) function [7]. These last functions, apparently little known to physicists, are thought to be the most natural generalizations of the exponential [7]. In the following we show that the states so defined satisfy the conditions (a)–(c), i.e. they are coherent states. Moreover, the resolution of the unity problem can be solved exactly, employing the techniques of Mellin and inverse Mellin transforms which we develop specially for this purpose. We shall pay particular attention to the question of uniqueness of the solutions. We next define the q-deformed ML functions $(0 < q \leqslant 1)$ and use them to produce q-deformed ML coherent states. Explicit calculations of properties like energy, statistics, fluctuations and the Heisenberg relations, reveal a highly interesting physical content. Finally, we address the question of eigenproperties of these states, and find explicitly the deformed boson annihilation operators of which the above states are eigenstates.

Necessary additional mathematical details and addenda are to be found in appendices A and B.

2. Coherent states via ML functions

In this paragraph we extend the construction of coherent states by choosing the coefficients $\rho(n)$ in

$$|z\rangle = \mathcal{N}^{-\frac{1}{2}}(|z|^2)\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{\rho(n)}}|n\rangle \tag{14}$$

with

$$\mathcal{N}(|z|^2) = \sum_{n=0}^{\infty}\frac{|z|^{2n}}{\rho(n)} \tag{15}$$

in such a way that the three requirements (a)–(c) are satisfied. One variant of this approach has been developed in [5], where the existence of the weight function was assumed and the $\rho(n)$ were given as its moments, i.e $\rho(n) = \int_0^\infty x^n W(x)\,\mathrm{d}x$, $n = 0, 1, 2, \ldots\infty$ with $\rho(0) = 1$. In the present work we adopt an inverse approach: we choose the $\rho(n)$ and then solve the moment problem and obtain $W(x)$. It should be noted that the conditions for the solubility of the moment problem are very restrictive: positive solution exists if and only if the two series of upper-left-corner determinants of the following Hankel–Hadamard (HH) matrices

$$\begin{aligned} h_0^{(n)}(i, j) &= \rho(i + j - 2) \\ h_1^{(n)}(i, j) &= \rho(i + j - 1) \end{aligned} \qquad (i, j = 1, 2, \ldots, n) \tag{16}$$

are positive for all $n = 1, 2, \ldots\infty$ [3]. For a general $\rho(n)$ the proof of positivity is virtually impossible. We therefore proceed differently by constructing a solution with the aid of the inverse Mellin transform. If such a construction leads to a positive solution the difficulty of proving the positivity of HH determinants has been overcome. Here we shall concentrate on the case where $\mathcal{N}(|z|^2)$ has an infinite radius of convergence, i.e. the label z runs over the whole complex plane. If $\rho(n) \neq n!$ the resulting normalization will be a function different from the exponential. Previous attemps in this spirit concentrated mainly on different kinds of generalized exponential functions [8]. In recent publications [6,9] we have chosen to solve certain functional equations and have ended up with another family of generalized exponentials permitting a solution of the moment problem.

Here we propose a simple ansatz for $\rho(n)$, which will lead naturally to a hitherto unexplored family of coherent states. To this end assume

$$\rho(n) = \frac{\Gamma(\alpha n + \beta)}{\Gamma(\beta)} \qquad \alpha, \beta > 0 \tag{17}$$

which leads to the normalized states

$$|z; \alpha, \beta\rangle = \left[\mathcal{N}_{\alpha,\beta}(|z|^2)\right]^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{z^n \sqrt{\Gamma(\beta)}}{\sqrt{\Gamma(\alpha n + \beta)}} |n\rangle \tag{18}$$

with normalization $\mathcal{N}_{\alpha,\beta}(|z|^2) = \Gamma(\beta) E_{\alpha,\beta}(|z|^2)$, where

$$E_{\alpha,\beta}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n + \beta)} \qquad \alpha, \beta > 0. \tag{19}$$

These last functions for $\beta = 1$ were introduced and analysed by Mittag-Leffler at the beginning of this century [10], and thus bear his name. We denote them $E_{\alpha,1}(x) \equiv E_\alpha(x)$. The generalized ML functions $E_{\alpha,\beta}(x)$ were introduced later [11,12]. For more recent references on this subject see [13,14,24]. These ML functions are natural generalizations of the exponential, and we shall refer to equation (18) as ML coherent states; the normalization is expressed through the generalized ML function in the same way as the normalization of standard coherent states is expressed by the exponential. The $E_{\alpha,\beta}(x)$ have only recently found application in physics [15]. We have summarized some of their properties and have given some examples for particular values of α and β in appendix A. It is intriguing to observe that for many different α, β, the ML functions $E_{\alpha,\beta}(x)$ can be expressed as combinations of elementary and special functions. We now investigate the states $|z; \alpha, \beta\rangle$ of equation (18) in the light of the requirements (a)–(c). The state $|z; \alpha, \beta\rangle$ is normalizable, since $E_{\alpha,\beta}(x)$ is an entire function and has no zeros for $x \geqslant 0$. The continuity in the label follows trivially. The overlap between two states is given by

$$\langle z; \alpha, \beta | z'; \alpha, \beta\rangle = \frac{E_{\alpha,\beta}(z^* z')}{\sqrt{E_{\alpha,\beta}(|z|^2) E_{\alpha,\beta}(|z'|^2)}} \tag{20}$$

with the numerator vanishing at the roots of $E_{\alpha,\beta}(x)$. $E_{\alpha,\beta}(x)$ do not have roots on the positive half-axis, as can be seen by applying the Hurwitz criterion [16] to equation (19). The states are orthogonal if $z^* z' = r_k(\alpha, \beta)$, where $r_k(\alpha, \beta)$ is the kth root of $E_{\alpha,\beta}(x)$. Since there is a countable number of roots, we conclude that two ML states are never orthogonal, except on a set of measure zero.

The resolution of unity leads to a requirement for $\tilde{W}_{\alpha,\beta}(|z|^2)$

$$\iint_{\mathbb{C}} \mathrm{d}^2 z \, |z; \alpha, \beta\rangle \tilde{W}_{\alpha,\beta}(|z|^2) \langle z; \alpha, \beta| = I \tag{21}$$

which, in turn, leads to the following Stieltjes power-moment problem ($x \equiv |z|^2$):

$$\int_0^\infty x^n W_{\alpha,\beta}(x) \, \mathrm{d}x = \frac{\Gamma(\alpha n + \beta)}{\Gamma(\beta)} \qquad n = 0, 1, 2 \ldots \tag{22}$$

where $W_{\alpha,\beta}(x) = \pi \tilde{W}_{\alpha,\beta}(x)/\mathcal{N}_{\alpha,\beta}(x) = \pi \tilde{W}_{\alpha,\beta}(x)/(\Gamma(\beta) E_{\alpha,\beta}(x))$. We find $W_{\alpha,\beta}(x)$ by interpreting (22) as the Mellin transform, for complex s, of $W_{\alpha,\beta}(x)$ [17]:

$$\int_0^\infty x^{s-1} W_{\alpha,\beta}(x) \, \mathrm{d}x = \frac{\Gamma(\alpha s + \beta - \alpha)}{\Gamma(\beta)} \tag{23}$$

or

$$W^{*}_{\alpha,\beta}(s) = \frac{\Gamma(\alpha s + \beta - \alpha)}{\Gamma(\beta)} \tag{24}$$

with $f^{*}(s)$ being the Mellin transform of $f(x)$, $\int_0^\infty x^{s-1} f(x)\,\mathrm{d}x = f^{*}(s) \equiv \mathcal{M}[f(x); s]$. To obtain $W_{\alpha,\beta}(x)$ we must perform an inverse Mellin transform on (24). We use $\mathcal{M}[x^b f(x^h); s] = \frac{1}{h} f^{*}(\frac{s+b}{h})$ (see equation (B.3)) with $h = \frac{1}{\alpha}$ and $b = \frac{\beta-\alpha}{\alpha}$ and thus

$$W_{\alpha,\beta}(x) = \frac{x^{\frac{\beta-\alpha}{\alpha}} \mathrm{e}^{-x^{\frac{1}{\alpha}}}}{\alpha\Gamma(\beta)} \qquad x > 0 \tag{25}$$

which is a positive solution of the Stieltjes moment problem. This solution is not unique in general, as may be ascertained from the Carleman condition (except for $\alpha < 2$, see section 3). For $\alpha \neq 1 \neq \beta$ we see that $W_{\alpha,\beta}(x)$ is a natural extension of $W_{1,1}(x) = \mathrm{e}^{-x}$. For completeness we note a few explicit examples of the normalizations $\mathcal{N}_{\alpha,\beta}(x)$ arising from some ML functions:

$$\begin{aligned}
\mathcal{N}_{1,2}(x) &= \frac{\mathrm{e}^{x}(1-\mathrm{e}^{-x})}{x} \\
\mathcal{N}_{1,3}(x) &= 2\left(\frac{\mathrm{e}^{x}(1-\mathrm{e}^{-x}(1+x))}{x^2}\right) \\
\mathcal{N}_{2,2}(x) &= \frac{\sinh(\sqrt{x})}{\sqrt{x}} \\
\mathcal{N}_{2,3}(x) &= 2\left(-\frac{1}{x} + \frac{\cosh(\sqrt{x})}{x}\right) \\
\mathcal{N}_{4,2}(x) &= \frac{1}{4x^{\frac{1}{4}}}\left(\mathrm{e}^{x^{\frac{1}{4}}} - \mathrm{e}^{-x^{\frac{1}{4}}} + 2\sin(x^{\frac{1}{4}})\right) \\
\mathcal{N}_{1,\frac{1}{2}}(x) &= \Gamma\left(\frac{1}{4}\right)\left(\frac{1}{\sqrt{\pi}} + \sqrt{x}\mathrm{e}^{x}\left(1-\mathrm{erfc}(\sqrt{x})\right)\right) \\
\mathcal{N}_{1,\frac{1}{4}}(x) &= \Gamma\left(\frac{1}{2}\right)\frac{\Gamma(\frac{3}{4})}{\sqrt{2}\pi}\left(1 + x^{\frac{3}{4}}\mathrm{e}^{x}\left(\frac{\pi\sqrt{2}}{\Gamma(\frac{3}{4})} - \Gamma(\frac{1}{4}, x)\right)\right) \text{ etc...}
\end{aligned} \tag{26}$$

where $\mathrm{erfc}(x)$ is the complementary error function and $\Gamma(\alpha, x)$ is the incomplete gamma function. All of these functions are equal to one at $x = 0$ and are increasing with positive x, in analogy to $\exp(x)$.

3. Constructing solutions of the Stieltjes moment problem

It has to be stressed that due to the intractability of verifying the positivity of the HH determinants (equation (16)), one is led to apply the inverse Mellin transformation directly to

$$\int_0^\infty x^n W(x)\,\mathrm{d}x = \rho(n) \qquad n = 0, 1, 2\ldots \tag{27}$$

or, equivalently, for s complex,

$$\int_0^\infty x^{s-1} W(x)\,\mathrm{d}x = \rho(s-1). \tag{28}$$

If $W(x) > 0$, then it is *a* solution of the Stieltjes moment problem, as in the case of equation (25). We call it a principal solution. Is this a unique solution? The answer is given by the (sufficient) condition of Carleman [3, 18]: if we know that a solution exists then

$$S \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \frac{1}{[\rho(n)]^{\frac{1}{2n}}} = \begin{cases} \infty & \text{the solution is unique} \\ < \infty & \text{non-unique solutions may exist.} \end{cases} \tag{29}$$

If $S < \infty$ one would like to know solutions other than the principal one. Returning to the ML moment problem, non-uniqueness may result if

$$S \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} a_n \equiv \sum_{n=1}^{\infty} [\Gamma(\beta)]^{\frac{1}{2n}} [\Gamma(\alpha n + \beta)]^{-\frac{1}{2n}} < \infty. \qquad (30)$$

A practical way to determine the possible convergence of the above S is to apply the logarithmic test [19]:

$$\text{if} \quad \lim_{n\to\infty} \frac{\ln(a_n)}{\ln(n)} < -1 \qquad S \text{ converges} \qquad (31)$$

$$\text{if} \quad \lim_{n\to\infty} \frac{\ln(a_n)}{\ln(n)} > -1 \qquad S \text{ diverges.} \qquad (32)$$

If, in equation (32), the limit is equal to -1, the test is inconclusive. It is convenient to use the asymptotic form of $\Gamma(x)$ for x large (real and positive), $\Gamma(x) \approx \sqrt{2\pi} x^{x-\frac{1}{2}} \mathrm{e}^{-x}$ (Stirling's formula). One then obtains $\lim_{n\to\infty} \ln(a_n)/\ln(n) = -\frac{\alpha}{2}$ and the logarithmic test gives

$\alpha > 2$	all	$\beta > 0$	S converges; non-unique solutions may exist
$\alpha < 2$	all	$\beta > 0$	S diverges; unique solution.

For $\alpha \leqslant 2$, all $\beta > 0$, $W_{\alpha,\beta}(x)$ of equation (25) is then the unique solution. How does one determine additional solutions for $\alpha > 2$, all $\beta > 0$? We know of no general method to answer this question, but propose here a construction based on further applications of the inverse Mellin transform which, for this given problem, will allow one to generate non-unique solutions. To be more specific, we will first construct, within the framework of ML functions, a set of functions $\omega_k(x)$ parametrized by k (to be defined later) such that *all* their moments vanish, i.e

$$\int_0^\infty x^n \omega_k(x)\,\mathrm{d}x = \int_0^\infty x^{s-1} \omega_k(x)\,\mathrm{d}x = 0 \qquad \begin{matrix} n = 0, 1, 2\ldots \\ s = 1, 2, 3\ldots. \end{matrix} \qquad (33)$$

Such functions play a fundamental role in the field of integral transforms [20].

The function $\omega_k(x)$ cannot be strictly positive as is seen by setting $n = 0$ in (33). By definition their Mellin transform $\omega_k^*(s)$ vanishes for $s = 1, 2, 3\ldots.$ We choose the following equations to produce the functions $\omega_k^*(s)$:

$$\int_0^\infty x^n \omega_k(x)\,\mathrm{d}x = \frac{\Gamma(\alpha n + \beta)}{\Gamma(\beta)} h_k(n) \qquad (34)$$

or, equivalently,

$$\int_0^\infty x^{s-1} \omega_k(x)\,\mathrm{d}x = \frac{\Gamma(\alpha s + \beta - \alpha)}{\Gamma(\beta)} h_k(s-1) \qquad (35)$$

where $h_k(n)$ is any holomorphic function vanishing for $n = 0, 1, 2\ldots$ (or $s = 1, 2, 3\ldots$). There is evidently an infinity of possible choices for $h_k(n)$. It turns out that the simplest choice $h_k(n) = \sin(\pi k(n+1))$ leads to a desired solution here. In other words,

$$\int_0^\infty x^{s-1} \omega_k(x)\,\mathrm{d}x = \frac{\Gamma(\alpha s + \beta - \alpha)}{\Gamma(\beta)} \sin(\pi k s) \qquad k = \pm 1, \pm 2, \pm 3\ldots \qquad (36)$$

and $\omega_k(x)$ is given by the inverse Mellin transform

$$\omega_k(x) = \frac{1}{2\pi \mathrm{i} \Gamma(\beta)} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \Gamma(\alpha s + \beta - \alpha) \sin(\pi k s) x^{-s}\,\mathrm{d}s \qquad k = \pm 1, \pm 2, \pm 3\ldots \qquad (37)$$

which is not listed in existing tables [19, pp 630–732, 21, 22]. An explicit calculation (see appendix B) gives

$$\omega_k(x) = \frac{x^{\frac{\beta-\alpha}{\alpha}} \exp(-x^{\frac{1}{\alpha}} \cos(\frac{\pi k}{\alpha}))}{\alpha\Gamma(\beta)} \sin\left(\pi k\left(1-\frac{\beta}{\alpha}\right) + x^{\frac{1}{\alpha}} \sin\left(\frac{\pi k}{\alpha}\right)\right) \qquad \alpha > 2|k|. \quad (38)$$

It is useful to recall the equation that $\omega_k(x)$ satisfies:

$$\int_0^\infty x^n \frac{x^{\frac{\beta-\alpha}{\alpha}} \exp(-x^{\frac{1}{\alpha}} \cos(\frac{\pi k}{\alpha}))}{\alpha\Gamma(\beta)} \sin\left(\pi k\left(1-\frac{\beta}{\alpha}\right) + x^{\frac{1}{\alpha}} \sin\left(\frac{\pi k}{\alpha}\right)\right) \mathrm{d}x = \frac{\Gamma(\alpha n+\beta)}{\Gamma(\beta)} \sin(\pi k(n+1)) \qquad \alpha > 2|k|. \quad (39)$$

Change the variable in equation (39): $y^{\frac{1}{\alpha}} = x^{\frac{1}{\alpha}} \cos(\frac{\pi k}{\alpha})$. Then equation (39) transforms into

$$\int_0^\infty x^n \frac{x^{\frac{\beta-\alpha}{\alpha}} \mathrm{e}^{-x^{\frac{1}{\alpha}}}}{\alpha\Gamma(\beta)} \sin\left(\pi k\left(1-\frac{\beta}{\alpha}\right) + x^{\frac{1}{\alpha}} \tan\left(\frac{\pi k}{\alpha}\right)\right) \mathrm{d}x = \cos\left(\frac{\pi k}{\alpha}\right)^{\alpha n+\beta} \frac{\Gamma(\alpha n+\beta)}{\Gamma(\beta)} \sin(\pi k(n+1)) \qquad \alpha > 2|k|. \quad (40)$$

Therefore, the function $\tilde{\omega}_k(x)$ defined by

$$\tilde{\omega}_k(x) = \frac{x^{\frac{\beta-\alpha}{\alpha}} \mathrm{e}^{-x^{\frac{1}{\alpha}}}}{\alpha\Gamma(\beta)} \sin\left(\pi k\left(1-\frac{\beta}{\alpha}\right) + x^{\frac{1}{\alpha}} \tan\left(\frac{\pi k}{\alpha}\right)\right) = W_{\alpha,\beta}(x) \sin\left(\pi k\left(1-\frac{\beta}{\alpha}\right) + x^{\frac{1}{\alpha}} \tan\left(\frac{\pi k}{\alpha}\right)\right) \quad (41)$$

has all its moments vanishing. It can now be added to the principal solution $W_{\alpha,\beta}(x)$ to produce a strictly positive function satisfying equation (22) giving a two-parameter (k, ϵ) family of non-unique solutions of equation (22), which we call $W(x; \alpha, \beta, k, \epsilon)$, and are given by

$$W(x; \alpha, \beta, k, \epsilon) = \frac{x^{\frac{\beta-\alpha}{\alpha}} \mathrm{e}^{-x^{\frac{1}{\alpha}}}}{\alpha\Gamma(\beta)} \left(1 + \epsilon \sin\left(\pi k\left(1-\frac{\beta}{\alpha}\right) + x^{\frac{1}{\alpha}} \tan\left(\frac{\pi k}{\alpha}\right)\right)\right) \qquad |\epsilon| < 1 \quad k = \pm1, \pm2, \pm3 \ldots \quad \alpha > 2|k|. \quad (42)$$

In figure 1 we represent the weight functions $W(x; \frac{14}{3}, \frac{31}{3}, 2, \frac{4}{5})$ of equation (42) and $W_{14/3,31/3}(x)$ of equation (25).

Choosing for $h_k(n)$ functions other than $\sin(\pi k(n+1))$ would give other families of solutions of equation (22), however at the cost of considerable complication in evaluating the inverse Mellin transform.

The weight function simplifies considerably if $\alpha = \beta$ and $k = 1$ $(\alpha > 2)$:

$$W(x; \alpha, \alpha, 1, \epsilon) = \frac{\mathrm{e}^{-x^{\frac{1}{\alpha}}}}{\alpha\Gamma(\beta)} \left(1 + \epsilon \sin\left(\tan\left(\frac{\pi}{\alpha}\right) x^{\frac{1}{\alpha}}\right)\right) \qquad |\epsilon| < 1 \quad \alpha > 2. \quad (43)$$

The case $\alpha = m$ (integer > 2) and $\beta = 1$ refers to the original ML function $E_m(x)$ (see equation (A.1)).

Obtaining equation (42) completes the demonstration that the ML-states of equation (18) possess a resolution of unity for $\alpha, \beta > 0$, with non-unique weight function for $\alpha > 2$.

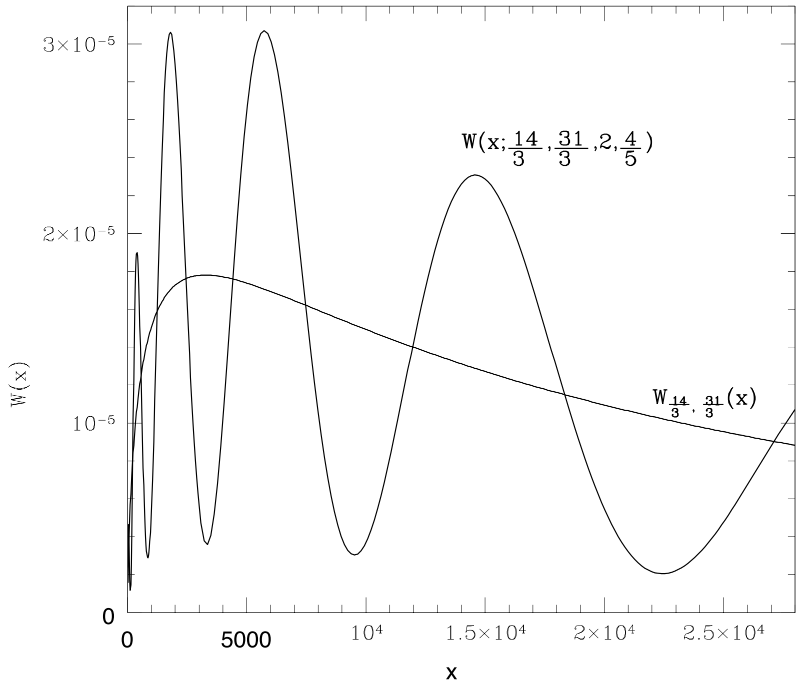

Figure 1. Weight function plots: the oscillating curve is the function given by equation (42) for the cases $\alpha = \frac{14}{3}$, $\beta = \frac{31}{3}$, $k = 2$ and $\epsilon = 0.8$, and the non-oscillating one represents the function given by equation (25) for the cases $\alpha = \frac{14}{3}$, $\beta = \frac{31}{3}$. By construction, the areas under these two curves are equal on $(0, \infty)$.

4. Coherent states via q-deformed ML functions

We now refer to the remarkable and simple relation satisfied by the ML functions of integer order m, ($m \geqslant 1$), see equation (A.2):

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m} E_m(x^m) = E_m(x^m), \qquad m = 1, 2, 3 \ldots . \qquad (44)$$
$$E_m(0) = 1.$$

For $m = 1$ this is simply the definition of $\exp(x)$. In the spirit of [6] we now construct two classes of q-deformed ($0 < q \leqslant 1$) ML functions $E_m(x; q)$ and $\varepsilon_m(x; q)$ which are defined as analytical solutions of the following functional equations ($m = 1, 2, 3, \ldots$):

$$\frac{\mathrm{d}^m E_m(z^m; q)}{\mathrm{d}z^m} = E_m((qz)^m; q) \qquad (45a)$$

$$E_m(0; q) = 1 \qquad (45b)$$

and

$$\frac{\mathrm{d}^m \varepsilon_m(z^m; q)}{\mathrm{d}z^m} = q^{\frac{m}{2}} \varepsilon_m((qz)^m; q) \qquad (46a)$$

$$\varepsilon_m(0; q) = 1. \qquad (46b)$$

Evidently, for $q = 1$, $E_m(z; 1) = \varepsilon_m(z; 1) = E_m(z)$. We recall that equations (45) are differential equations with the argument shifted by q. Consequently, there are no continuity arguments which would permit one to deduce any property of their solutions for the case $q = 1$ from, say, $q = 0.99$.

We will find the solutions of equations (45) and (46) assuming that the functions $E_m(z; q)$ and $\varepsilon_m(z; q)$ can be expressed as power series:

$$E_m(z; q) = \sum_{n=0}^{\infty} a_n(m, q) z^n \tag{47}$$

$$\varepsilon_m(z; q) = \sum_{n=0}^{\infty} b_n(m, q) z^n. \tag{48}$$

We substitute equation (47) into (45) and (48) into (46) and obtain the linear recursion relations for the coefficients $a_n(m, q)$ and $b_n(m, q)$:

$$a_{n+1}(m, q) = \frac{a_n(m, q) q^{mn}}{\prod_{l=1}^{m}(mn + l)} \qquad n = 0, 1, 2 \ldots \tag{49a}$$

$$a_0(m, q) = 1 \qquad m = 1, 2 \ldots \tag{49b}$$

$$b_{n+1}(m, q) = \frac{b_n(m, q) q^{m(n+\frac{1}{2})}}{\prod_{l=1}^{m}(mn + l)} \qquad n = 0, 1, 2 \ldots \tag{50a}$$

$$b_0(m, q) = 1 \qquad m = 1, 2 \ldots \tag{50b}$$

from which the explicit forms of the two required q-deformed ML functions follow:

$$E_m(z; q) = \sum_{n=0}^{\infty} \frac{q^{\frac{mn(n-1)}{2}}}{\Gamma(mn + 1)} z^n \qquad m = 1, 2 \ldots \tag{51}$$

$$\varepsilon_m(z; q) = \sum_{n=0}^{\infty} \frac{q^{\frac{mn^2}{2}}}{\Gamma(mn + 1)} z^n \qquad m = 1, 2 \ldots. \tag{52}$$

They are convergent in the whole complex plane. We now construct new normalized states using the coefficients of the above functions:

$$|z; m, q\rangle_1 = \frac{1}{\sqrt{E_m(|z|^2; q^2)}} \sum_{n=0}^{\infty} \frac{q^{\frac{mn(n-1)}{2}}}{\sqrt{\Gamma(mn + 1)}} z^n |n\rangle \tag{53}$$

$$|z; m, q\rangle_2 = \frac{1}{\sqrt{\varepsilon_m(|z|^2; q^2)}} \sum_{n=0}^{\infty} \frac{q^{\frac{mn^2}{2}}}{\sqrt{\Gamma(mn + 1)}} z^n |n\rangle. \tag{54}$$

The overlaps are, respectively,

$${}_1\langle z; m, q | z'; m, q\rangle_1 = \frac{E_m(z^* z'; q^2)}{\sqrt{E_m(|z|^2; q^2) E_m(|z'|^2; q^2)}} \tag{55}$$

$${}_2\langle z; m, q | z'; m, q\rangle_2 = \frac{\varepsilon_m(z^* z'; q^2)}{\sqrt{\varepsilon_m(|z|^2; q^2) \varepsilon_m(|z'|^2; q^2)}}. \tag{56}$$

The remarks following equation (20) apply here too. The resolution of unity in terms of $|z; m, q\rangle_{1,2}$ reads as

$$\iint_{\mathbb{C}} \mathrm{d}^2 z \, |z; m, q\rangle_{1,2} \tilde{W}_{1,2}(|z|^2; m, q) \, {}_{1,2}\langle z; m, q| = I \tag{57}$$

where the positive weight functions $\tilde{W}_{1,2}(|z|^2; m, q)$, yet to be determined, are the solutions of the following Stieltjes moment problems:

$$\pi \int_0^{\infty} x^n \frac{\tilde{W}_1(x; m, q)}{E_m(x, q^2)} \, \mathrm{d}x = \frac{\Gamma(mn + 1)}{q^{mn(n-1)}} \qquad \begin{array}{l} n = 0, 1, 2 \ldots \\ m = 1, 2, 3 \ldots \end{array} \tag{58}$$

$$\pi \int_0^{\infty} x^n \frac{\tilde{W}_2(x; m, q)}{\varepsilon_m(x, q^2)} \, \mathrm{d}x = \frac{\Gamma(mn + 1)}{q^{mn^2}} \qquad \begin{array}{l} n = 0, 1, 2 \ldots \\ m = 1, 2, 3 \ldots. \end{array} \tag{59}$$

Application of the Carleman criterion (29) leads to the following:

- if $q < 1$ the solutions are possibly non-unique for all m;
- if $q = 1$ the solution is unique for $m = 1, 2$ and possibly non-unique for $m = 3, 4, 5 \ldots \infty$.

For $W_1(x; m, q) = \pi \tilde{W}_1(x; m, q)/E_m(x, q^2)$ and $W_2(x; m, q) = \pi \tilde{W}_2(x; m, q)/\varepsilon_m(x, q^2)$, equations (58), and (59) read

$$\int_0^\infty x^n W_1(x; m, q)\,\mathrm{d}x = \frac{\Gamma(mn+1)}{q^{mn(n-1)}} \qquad \begin{array}{l} n = 0, 1, 2 \ldots \\ m = 1, 2, 3 \ldots \end{array} \tag{60}$$

$$\int_0^\infty x^n W_2(x; m, q)\,\mathrm{d}x = \frac{\Gamma(mn+1)}{q^{mn^2}} \qquad \begin{array}{l} n = 0, 1, 2 \ldots \\ m = 1, 2, 3 \ldots . \end{array} \tag{61}$$

The solutions $W_1(x; m, q)$ and $W_2(x; m, q)$ are related through the change of variable $x = q^m y$ in equation (60) which leads to

$$\int_0^\infty y^n \frac{W_1(q^m y; m, q)}{q^{-m}}\,\mathrm{d}y = \frac{\Gamma(mn+1)}{q^{mn^2}}. \tag{62}$$

This tells us that if $W_1(x; m, q)$ satisfies equation (60) then $W_2(x; m, q) = q^m W_1(q^m x; m, q)$ satisfies equation (61). In other words, we have the following scaling relation:

$$W_1(x; m, q) = q^{-m} W_2(q^{-m} x; m, q) \tag{63}$$

and it is sufficient to solve equation (58) only.

We first solve two simpler problems

$$\int_0^\infty x^n \phi_m(x)\,\mathrm{d}x = \Gamma(mn+1) \qquad n = 0, 1, 2 \ldots \tag{64}$$

and

$$\int_0^\infty x^n \psi(x, q)\,\mathrm{d}x = \frac{1}{q^{mn^2}} \qquad n = 0, 1, 2 \ldots . \tag{65}$$

Through equation (25)

$$\phi_m(x) = \frac{x^{\frac{1-m}{m}} \mathrm{e}^{-x^{\frac{1}{m}}}}{m}. \tag{66}$$

Rewrite equation (65) as:

$$\int_0^\infty x^{s-1} \psi(x, q)\,\mathrm{d}x = \mathrm{e}^{\delta(s-1)^2} \tag{67}$$

where $\delta = m \ln(\frac{1}{q}) > 0$ and note that a solution to an auxiliary problem

$$\int_0^\infty x^{s-1} \psi_0(x, q)\,\mathrm{d}x = \mathrm{e}^{\delta s^2} \tag{68}$$

is given by [21]:

$$\psi_0(x, q) = \frac{1}{2\sqrt{\pi m \ln(\frac{1}{q})}} \mathrm{e}^{-\frac{(\ln(x))^2}{4\delta}}. \tag{69}$$

We now use equations (68) and (B.3) to obtain the solution of equation (65) as $\psi(x, q) = x^{-1}\psi_0(x, q)$ with

$$\psi(x, q) = \frac{1}{2x\sqrt{\pi\delta}} \mathrm{e}^{-\frac{(\ln(x))^2}{4\delta}} \tag{70}$$

which is the lognormal distribution function [23]

$$p(\mu, \sigma, x) = \frac{1}{\sigma x\sqrt{2\pi}} \exp\left(-\frac{(\ln(x) - \mu)^2}{2\sigma^2}\right) \tag{71}$$

with $\mu = 0$ and $\sigma^2 = 2\delta$. Finally, since from equation (64)

$$\int_0^\infty x^{s-1}\phi_m(x)\,\mathrm{d}x = \Gamma(m(s-1)+1) \equiv \phi_m^*(s) \tag{72}$$

and from equation (65)

$$\int_0^\infty x^{s-1}\psi(x, q)\,\mathrm{d}x = q^{-m(s-1)^2} \equiv \psi^*(s, q) \tag{73}$$

we can use the convolution property of the Mellin transform (Parseval equation), see equation (B.6), to obtain

$$\begin{aligned} W_2(x; m, q) &= \int_0^\infty \psi_m\left(\frac{x}{u}, q\right)\phi(u)\frac{1}{u}\,\mathrm{d}u \\ &= \frac{1}{2m\sqrt{\pi\delta x}}\int_0^\infty u^{\frac{1-m}{m}}\mathrm{e}^{-u^{\frac{1}{m}}}\exp\left(-\frac{(\ln(\frac{x}{u}))^2}{4\delta}\right)\mathrm{d}u \end{aligned} \tag{74}$$

and with equation (63)

$$W_1(x; m, q) = \frac{1}{2m\sqrt{\pi\delta x}}\int_0^\infty u^{\frac{1-m}{m}}\mathrm{e}^{-u^{\frac{1}{m}}}\exp\left(-\frac{(\ln(\frac{x}{q^m u}))^2}{4\delta}\right)\mathrm{d}u. \tag{75}$$

Both W_1 and W_2 are positive functions of x for $0 \leqslant x \leqslant \infty$ and are thus principal solutions of the moment problems (60) and (61). We illustrate this by displaying $W_1(x; 3, \frac{1}{2})$ as a function of x in figure 2.

The non-unique solutions can be obtained by using the methods of the previous section but they will not be reproduced here.

The states (53) and (54) possess a resolution of unity in terms of positive functions $W_{1,2}(x; m, q)$ and are thus coherent states.

5. Physical applications

The ML and q-deformed ML coherent states belong to a category of states which can be expressed in the form

$$|z\rangle = \mathcal{N}^{-\frac{1}{2}}(|z|^2)\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{\rho(n)}}|n\rangle \tag{76}$$

with the normalization

$$\mathcal{N}(|z|^2) = \sum_{n=0}^{\infty}\frac{|z|^{2n}}{\rho(n)} \tag{77}$$

which is an entire function. In equations (76) and (77) the $\rho(n)$ are the nth moments of a positive weight function in the resolution of unity. As we have seen above, in a general case the weight function is not unique. The simple structure of these equations allows relatively straightforward calculations of various expectation values in $|z\rangle$, which by necessity are functions of the set of $\rho(n)$; in some cases (see below) these are functions only of normalization.

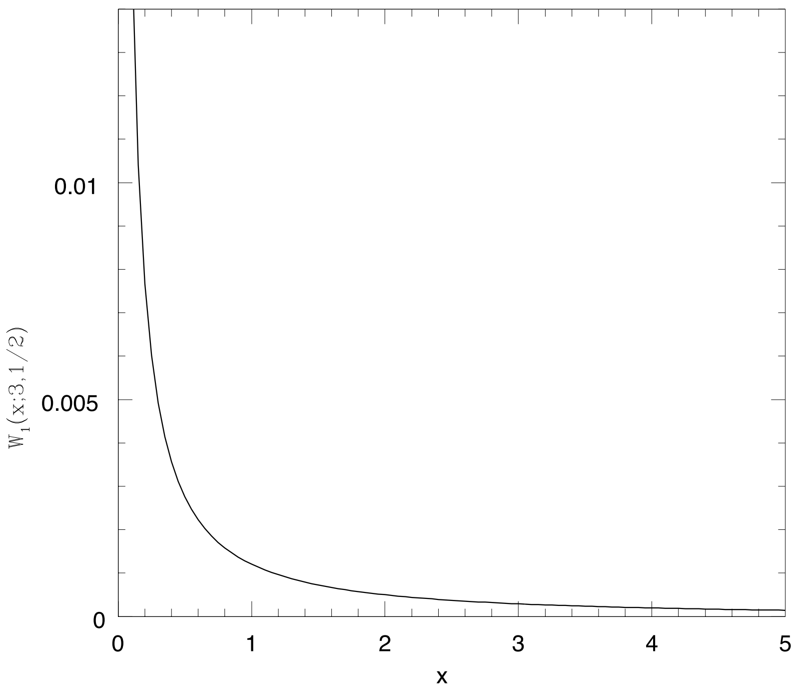

Figure 2. Plot of weight function $W_1(x; m, q)$ for $m = 3$ and $q = \frac{1}{2}$ as a function of x (see equation (75)).

Some expectation values of polynomial Hermitian operators are expressible through the derivatives of $\mathcal{N}(|z|^2)$, for r integer

$$\langle z| \left(\hat{a}^\dagger\right)^r \hat{a}^r |z\rangle = \frac{|z|^{2r}}{\mathcal{N}(|z|^2)} \left(\frac{\mathrm{d}}{\mathrm{d}|z|^2}\right)^r \mathcal{N}(|z|^2) \qquad r = 0, 1, 2 \ldots. \tag{78}$$

This means that the normalization of $|z\rangle$ is a generating function for certain expectation values in $|z\rangle$. We quote for completeness a generalization of equation (78) for r, $p = 0, 1, 2\ldots$:

$$\langle z|(\hat{a}^\dagger)^p \hat{a}^r|z\rangle = \frac{(z^*)^p z^r}{\mathcal{N}(|z|^2)} \sum_{n=0}^{\infty} \left[\frac{(n+p)!(n+r)!}{\rho(n+p)\rho(n+r)}\right]^{\frac{1}{2}} \frac{|z|^{2n}}{n!} \qquad \begin{array}{l} r = 0, 1, 2\ldots \\ p = 0, 1, 2\ldots \end{array} \tag{79}$$

which, if $r \neq p$ includes non-Hermitian operators. The formulae (78) and (79) have been used to calculate various expectation values and fluctuations of physical quantities in the ML and q-deformed ML states.

From equation (76) it is easy to determine the probability of finding the state $|n\rangle$ in the state $|z; \alpha, \beta\rangle$. It is equal to ($x \equiv |z|^2$)

$$p_{\alpha,\beta}(n, x) = \frac{x^n \Gamma(\beta)}{E_{\alpha,\beta}(x)\Gamma(\alpha n + \beta)} \tag{80}$$

and is different from the Poisson distribution characterizing the conventional coherent state unless $\alpha = \beta = 1$. Further information about inherent statistical characteristics may be obtained from the Mandel parameter Q_M [29] ($\hat{N} = \hat{a}^\dagger\hat{a}$):

$$Q_M = \frac{(\Delta\hat{N})^2 - \langle\hat{N}\rangle}{\langle\hat{N}\rangle} \tag{81}$$

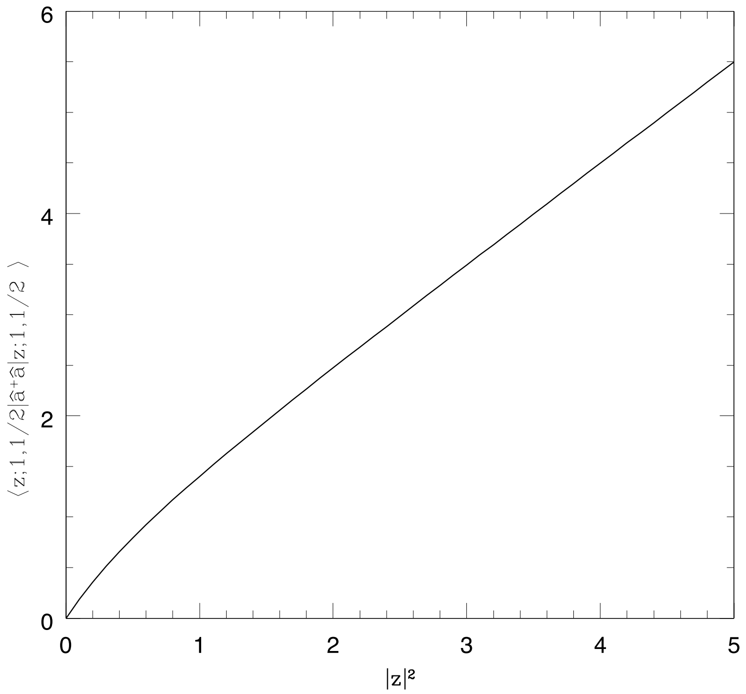

Figure 3. Plot of $\langle z; 1, \frac{1}{2}|\hat{a}^\dagger\hat{a}|z; 1, \frac{1}{2}\rangle$ as a function of $|z|^2$. This expectation value of $\hat{a}^\dagger\hat{a}$ may be regarded as the energy, proportional to the number of photons in the state $|z; 1, \frac{1}{2}\rangle$.

where $(\Delta\hat{A})^2 \equiv \langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2$. All averages are taken in the state $|z; \alpha, \beta\rangle$. For illustrative purposes we shall limit ourselves here to the calculations for $\alpha = 1$, $\beta = \frac{1}{2}$, that is for the moments $\rho(n) = \Gamma(n+\frac{1}{2})/\Gamma(\frac{1}{2})$, $n = 0, 1, 2, \ldots, \infty$. In figure 3 we represent the expectation value $\langle z; 1, \frac{1}{2}|\hat{N}|z; 1, \frac{1}{2}\rangle$ as a function of $x = |z|^2$. We notice that this energy is larger that the value (x) obtained in the conventional coherent states.

In figure 4 we plot the Mandel parameter Q_M which is negative, and consequently the state $|z; 1, \frac{1}{2}\rangle$ is sub-Poissonian. We next turn to the expectation values of momentum P and the coordinate Q in $|z; 1, \frac{1}{2}\rangle$. In figure 5 we plot $(\Delta\hat{Q})^2$ which for real z is less than 0.5, thus giving squeezing in this quadrature. In figure 6 we show $(\Delta\hat{P})^2$ which is squeezed for purely imaginary z. Results for other values of α and β may be obtained in a similar way.

In figure 7 we present the variance of the coordinate Q in the q-deformed ML state $|z; 2, q\rangle_1$ (see equation (53)) for several values of q. We observe that this state is always squeezed for small x but becomes dilated above a certain value of x which depends on q.

Complete exploration of other values of m, q and x confirms the rich variety of behaviour observed for $m = 2$.

6. ML states and boson deformations

The conventional coherent state $|z\rangle$ is an eigenstate of the annihilation operator $\hat{a}$, $\hat{a}|z\rangle = z|z\rangle$. In this section we show that the ML states are themselves eigenstates of a *deformed* boson operator $\hat{b}$, which is a *deformation* of $\hat{a}$. This observation permits us to situate our work in the context of *deformed bosons*, a subject thoroughly investigated recently [27]. It turns out that

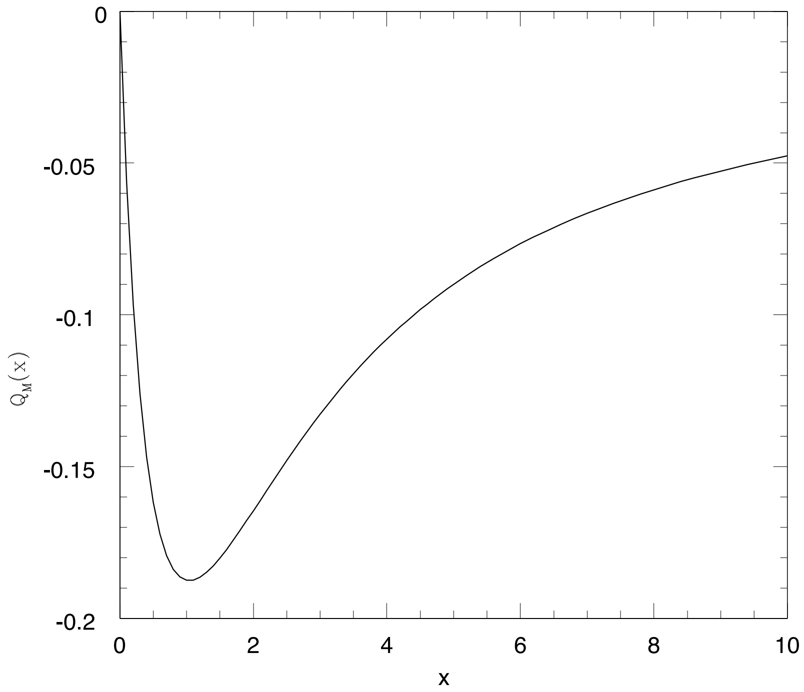

Figure 4. Plot of the Mandel parameter $Q_M(x)$ as a function of $x = |z|^2$. Since $Q_M(x) < 0$ for all $x > 0$, the state $|z; 1, \frac{1}{2}\rangle$ is a sub-Poissonian one.

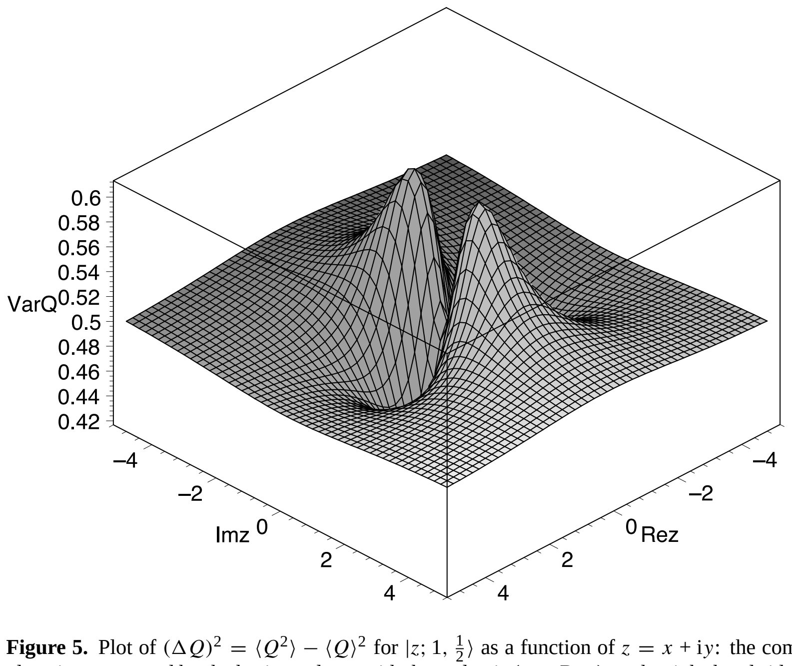

Figure 5. Plot of $(\Delta Q)^2 = \langle Q^2\rangle - \langle Q\rangle^2$ for $|z; 1, \frac{1}{2}\rangle$ as a function of $z = x + \mathrm{i}y$: the complex plane is represented by the horizontal one with the real axis ($x = \mathrm{Re}\, z$) on the right-hand side, and the imaginary axis ($y = \mathrm{Im}\, z$) on the left-hand side.

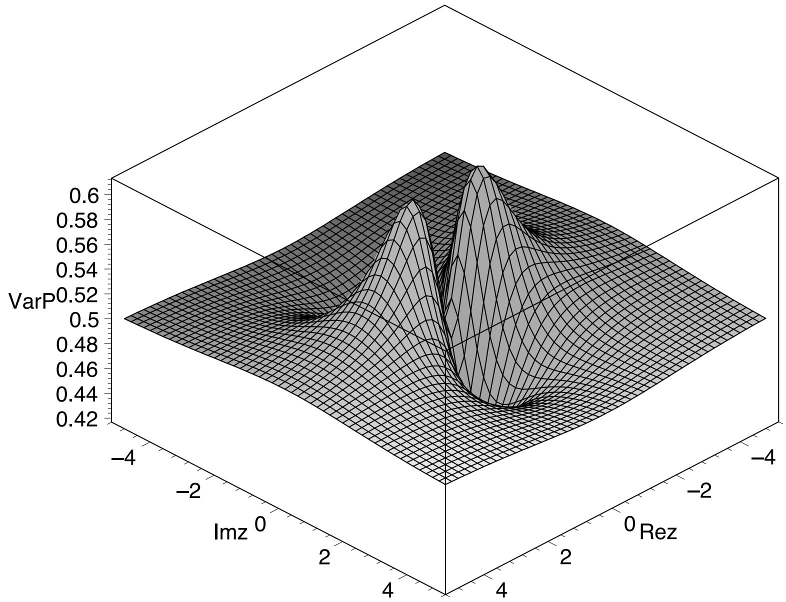

Figure 6. Plot of $(\Delta P)^2 = \langle P^2\rangle - \langle P\rangle^2$ for $|z; 1, \frac{1}{2}\rangle$ as a function of $z = x + \mathrm{i}y$: the complex plane is represented by the horizontal one with the real axis ($x = \mathrm{Re}\, z$) on the right-hand side, and the imaginary axis ($y = \mathrm{Im}\, z$) on the left-hand side.

an auxiliary notion of a *box* function $[x]_{\alpha,\beta}$ ('box' x) will be helpful [28], with α, β referring to appropriate ML functions. The box function will be used to define the generalized exponential function, identified here with the ML function, (for the q-deformed ML, see later) as

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_{\alpha,\beta}!} \tag{82}$$

with

$$[n]_{\alpha,\beta}! \stackrel{\text{def}}{=} [1]_{\alpha,\beta}[2]_{\alpha,\beta} \ldots [n]_{\alpha,\beta} = \Gamma(\alpha n + \beta). \tag{83}$$

The deformation of the boson operator $\hat{a}$ is determined by $[n]$, leading to the deformed commutator [28]

$$\hat{b}_{\alpha,\beta}\hat{b}^{\dagger}_{\alpha,\beta} - \hat{b}^{\dagger}_{\alpha,\beta}\hat{b}_{\alpha,\beta} = [\hat{N}+1]_{\alpha,\beta} - [\hat{N}]_{\alpha,\beta} \tag{84}$$

where $\hat{N} = \hat{a}^{\dagger}\hat{a}$, $\hat{N}|n\rangle = n|n\rangle$. From (83) it follows that it is natural to define

$$[n]_{\alpha,\beta} \equiv \frac{[n]_{\alpha,\beta}!}{[n-1]_{\alpha,\beta}!} = \frac{\Gamma(\alpha n + \beta)}{\Gamma(\alpha(n-1)+\beta)} \tag{85}$$

if $n > 0$, and $[0]_{\alpha,\beta} = 0$. The operators $\hat{b}$ and $\hat{a}$ act on the same Fock space $\{|n\rangle\}$. Then from equation (84) we obtain

$$\hat{b}_{\alpha,\beta}|n\rangle = \sqrt{[n]_{\alpha,\beta}}|n-1\rangle \tag{86}$$

$$\hat{b}^{\dagger}_{\alpha,\beta}|n\rangle = \sqrt{[n+1]_{\alpha,\beta}}|n+1\rangle \tag{87}$$

which give the explicit relation between $\hat{a}$ and $\hat{b}$

$$\hat{b}^{\dagger}_{\alpha,\beta} = \hat{a}^{\dagger}\sqrt{\frac{[\hat{N}+1]_{\alpha,\beta}}{\hat{N}+1}}. \tag{88}$$

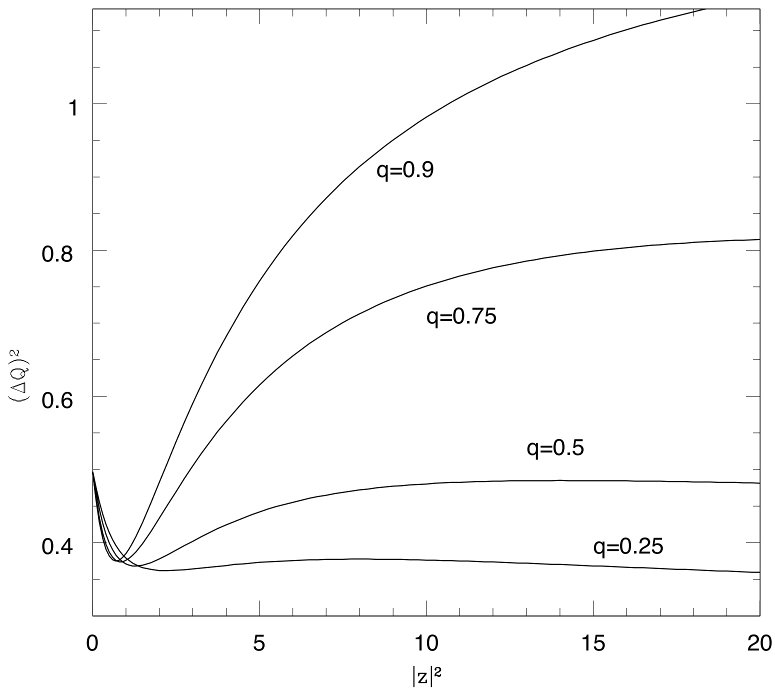

Figure 7. Plot of $(\Delta Q)^2 = \langle Q^2\rangle - \langle Q\rangle^2$ as a function of $|z|^2$ for several values of q and $m = 2$. Here, the averages are calculated in the states $|z; m, q\rangle_1$ (given by equation (53)); we thus have $\langle Q\rangle \equiv {}_1\langle z; m, q|Q|z; m, q\rangle_1$.

Using the above relations we rewrite the state $|z; \alpha, \beta\rangle$ as

$$|z; \alpha, \beta\rangle = [E_{\alpha,\beta}(|z|^2)]^{-\frac{1}{2}} E_{\alpha,\beta}(z\hat{b}^\dagger_{\alpha,\beta})|0\rangle \tag{89}$$

$$= [E_{\alpha,\beta}(|z|^2)]^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{[n]_{\alpha,\beta}!}}|n\rangle \tag{90}$$

and, from equation (86), the eigenproperty holds

$$\hat{b}_{\alpha,\beta}|z; \alpha, \beta\rangle = z|z; \alpha, \beta\rangle. \tag{91}$$

This formalism generalizes easily to q-deformed ML functions, $E_m(x; q)$ and $\varepsilon_m(x; q)$. We carry it through for $E_m(x; q)$ only:

$$|z; m, q\rangle_1 = [E_m(|z|^2; q^2)]^{-\frac{1}{2}} E_m(z\hat{b}^\dagger_{m;q}; q)|0\rangle \tag{92}$$

$$= [E_m(|z|^2; q^2)]^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{[n]_{m;q}!}}|n\rangle \tag{93}$$

with

$$[n]_{m;q}! \equiv \frac{\Gamma(mn+1)}{q^{mn(n-1)}} \qquad m, n = 1, 2, 3\ldots \tag{94}$$

and

$$[n]_{m;q} \equiv \frac{[n]_{m;q}!}{[n-1]_{m;q}!} = \begin{cases} \dfrac{(mn)!}{(m(n-1))!}\dfrac{1}{q^{2m(n-1)}} & n > 0 \\ 0 & n = 0. \end{cases} \tag{95}$$

It is instructive to calculate the explicit deformations of the commutator equation (84) and of the operator itself equation (88), based on $\hat{b}_{m;q}\hat{b}^{\dagger}_{m;q} - \hat{b}^{\dagger}_{m;q}\hat{b}_{m;q} = [\hat{N}+1]_{m;q} - [\hat{N}]_{m;q}$:

$$m = 1 \qquad [n]_{1;q} = \frac{n}{q^{2(n-1)}} \tag{96}$$

$$\hat{b}_{1;q}\hat{b}^{\dagger}_{1;q} - q^{-2}\hat{b}^{\dagger}_{1;q}\hat{b}_{1;q} = q^{-2\hat{N}} \tag{97}$$

$$\hat{b}_{1;q} = \mathrm{e}^{\ln(1/q)\hat{N}}\hat{a} \tag{98}$$

$$m = 2 \qquad [n]_{2;q} = \frac{2n(2n-1)}{q^{4(n-1)}} \tag{99}$$

$$\hat{b}_{2;q}\hat{b}^{\dagger}_{2;q} - q^{-4}\hat{b}^{\dagger}_{2;q}\hat{b}_{2;q} = \frac{8\hat{N}+2}{q^{4\hat{N}}} \tag{100}$$

$$\hat{b}_{2;q} = \sqrt{4\hat{N}+2}\,\mathrm{e}^{2\hat{N}\ln(1/q)}\hat{a}. \tag{101}$$

The results for $m > 2$ are readily obtained. Note that for $m = 1$ the formulae are derived in [6]. In this case only, when $q = 1$, then $\hat{b}_{1;1} \equiv \hat{a}$. For $m \geqslant 2$ we have a deformation even for $q = 1$: we still have $\hat{b}_{m;1} \neq \hat{a}$. For $m = 2$, $q = 1$ we obtain a realization of the $SU(1,1)$ algebra; when $q \neq 1$ we obtain a deformation of that algebra, different from the standard deformation [34]. For $|z; m, q\rangle_1$ of equation (92) the eigenproperty

$$\hat{b}_{m;q}|z; m, q\rangle_1 = z|z; m, q\rangle_1 \tag{102}$$

still holds.

Note that equations (97) and (100) are new deformations of the boson commutator which differ from the standard ones [30–33].

7. Conclusions

Coherent states provide an important theoretical resource for the description of physical phenomena. Thus the standard (Glauber) coherent states describe an ideal laser in quantum optics, a good approximation to a real laser. More recently, these same states have been used to provide a good starting point for the description of a boson condensate in the phenomenon of Bose–Einstein condensation. However, the repertoire of coherent states has been rather restricted, usually limited to those states associated with Lie groups or, more recently, quantum groups. Starting from the axiomatic description of coherent states, one opens up the possibilities limitlessly; however, the difficulty remains of demonstrating the resolution of the identity property for such putative coherent states. In this paper we illustrated methods for the solution of this problem by the use of the Mellin and inverse Mellin transforms, with specific application to a new family of coherent states based on the functions of Mittag-Leffler, natural generalizations of the exponential function which plays a pivotal role in the standard case. Since the ML functions are often related to well known elementary classical functions of analysis, it is not surprising that the associated coherent states may well prove important in the discussion of physical phenomena; thus, squeezed vacuum states are related to the function $\cosh(x) = E_2(x^2)$. In this paper we examined the physical implications of the ML states; in particular their squeezing properties. Finally, we showed that this family of states, and their q-deformations, can be expressed as eigenstates of deformed boson operators, thus demonstrating the relationship of the present work with the study of deformed bosons.

Acknowledgments

We thank L Haddad for an important contribution to this work, J R Klauder and J P Gazeau for illuminating discussions. We thank R Gorenflo for informing us about [13, 15, 24].

Appendix A. ML functions and some of their properties

The ML functions were introduced in [10] and are defined by the series

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} \qquad \alpha > 0 \quad z \in \mathbb{C}. \tag{A.1}$$

In our considerations we use only the values $\alpha > 0$, for which $E_\alpha(z)$ is an entire function. A rather complete account of the properties of $E_\alpha(z)$ can be found in [7]. A more recent detailed exposition with an actualized reference list is the [24].

For m integer ($m > 0$), $E_m(z)$ satisfy

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^m E_m(z^m) = E_m(z^m) \tag{A.2}$$

from which we see clearly that $E_m(z^m)$ are generalizations of the exponential ($E_1(z) \equiv \mathrm{e}^z$) and $\cosh(z)$ ($E_2(z^2) \equiv \cosh(z)$). Generalized ML functions are defined as

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} \qquad \alpha, \beta > 0 \quad z \in \mathbb{C} \tag{A.3}$$

and $E_{\alpha,1}(z) \equiv E_\alpha(z)$.

Since $E_{\alpha,\beta}(z)$ are proportional to the normalization of coherent states $|z; \alpha, \beta\rangle$ it is essential to verify that $E_{\alpha,\beta}(|z|^2)$ is never zero. In fact, we have the following statements about the zeros of ML functions:

- $E_\alpha(z)$ for $\alpha > 0$ has no zeros on the positive real axis (for $\alpha = 1$ there are no zeros at all);
- $E_\alpha(z)$ for $\alpha \geqslant 2$ has a countably infinite number of zeros on the real negative axis and no other zeros;
- $E_{\alpha,\beta}(z)$ for $\alpha, \beta > 0$ has no zeros on the positive real axis.

Here we quote some examples of $E_{\alpha,\beta}(z)$ and their relations to other elementary and special functions:

$$E_1(z) = \mathrm{e}^z$$
$$E_2(z) = \cosh(\sqrt{z}),$$
$$E_3(z) = \frac{1}{3}\left(\mathrm{e}^{z^{1/3}} + 2\mathrm{e}^{-\frac{z^{1/3}}{2}} \cos\left(\frac{\sqrt{3}}{2} z^{1/3}\right)\right)$$
$$E_4(z) = \frac{1}{4}\left(\mathrm{e}^{z^{1/4}} + \mathrm{e}^{-z^{1/4}}\right) + \frac{1}{2}\cos(z^{\frac{1}{4}}) \text{ etc} \ldots$$
$$E_{1,2}(z) = \frac{\mathrm{e}^z(1 - \mathrm{e}^{-z})}{z}$$
$$E_{1,3}(z) = \frac{\mathrm{e}^z(1 - \mathrm{e}^{-z}(1 + z))}{z^2}$$
$$E_{2,2}(z) = \frac{\sinh(\sqrt{z})}{\sqrt{z}}$$

$$E_{2,3}(z) = \frac{1}{2z}\left(-1+\cos(\sqrt{z})\right) \text{ etc}\ldots$$
$$E_{1,\frac{3}{2}}(z) = \frac{\mathrm{e}^{z}(1-\mathrm{erfc}(\sqrt{z}))}{\sqrt{z}}$$
$$E_{2,\frac{3}{2}}(z) = \frac{1}{2}\frac{\mathrm{e}^{\sqrt{z}}\,\mathrm{erf}(z^{1/4}) + \mathrm{e}^{-\sqrt{z}}\,\mathrm{erfi}(z^{1/4})}{z^{1/4}}$$
$$E_{3,\frac{1}{2}}(z) = \frac{1}{3}\frac{3+\sqrt{3}z^{1/6}\sqrt{\pi}\mathrm{e}^{\frac{z^{1/3}}{3}}\,\mathrm{erf}\left(\frac{z^{1/6}}{\sqrt{3}}\right)}{\pi} \text{ etc}\ldots$$

where $\mathrm{erf}(x)$ is the error function, $\mathrm{erfc}(x)$ is the complementary error function and $\mathrm{erfi}(x) = \mathrm{i}\,\mathrm{erf}(\mathrm{i}x)$ is the imaginary error function.

Appendix B. Mellin and inverse Mellin transforms and their applications to the Stieltjes moment problem

We quote here the main formulae of the Mellin and inverse Mellin transform [17] as applied to the solution of the moment problem.

The integral

$$\int_0^\infty x^{s-1} f(x)\,\mathrm{d}x \equiv f^*(s) \stackrel{\text{def}}{=} \mathcal{M}[f(x); s] \tag{B.1}$$

is called the Mellin transform, for complex s, of the function $f(x)$ and is denoted $\mathcal{M}[f(x); s] = f^*(s)$. The inversion of equation (B.1) determines $f(x)$ in terms of $f^*(s)$ and is denoted $f(x) = \mathcal{M}^{-1}[f^*(s); x]$. The explicit formula for obtaining $f(x)$ from $f^*(s)$ is given by

$$f(x) = \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} f^*(s)x^{-s}\,\mathrm{d}s \stackrel{\text{def}}{=} \mathcal{M}^{-1}[f^*(s); x] \tag{B.2}$$

which denotes the inverse Mellin transform. The conditions on $f^*(s)$ which ensure that the inversion equation (B.2) exist are enumerated in [17]. In fact, for a vast class of functions $f^*(s)$, the value of $f(x)$ does not depend on the value of c and one can fix $c = 0$, which effectively means integration over the imaginary axis. The Mellin transform obeys a certain number of relations obtained from a linear transformation of s, $(a, b, h \in \mathbb{R}; a, h > 0)$:

$$\mathcal{M}[x^b f(ax^h); s] = \frac{1}{h}a^{-\frac{s+b}{h}} f^*\left(\frac{s+b}{h}\right) \tag{B.3}$$
$$\mathcal{M}[x^b f(ax^{-h}); s] = \frac{1}{h}a^{\frac{s+b}{h}} f^*\left(-\frac{s+b}{h}\right). \tag{B.4}$$

For our applications the essential property is the so-called Mellin convolution (which has to be distinguished from a conventional convolution of, say, Fourier transforms). For arbitrary a, b the following indentities are valid:

$$x^a\int_0^\infty t^b f(tx)g(t)\,\mathrm{d}t = \frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} f^*(s+a)g^*(1-s-a+b)x^{-s}\,\mathrm{d}s \tag{B.5}$$
$$x^a\int_0^\infty t^b f\left(\frac{x}{t}\right)g(t)\,\mathrm{d}t = \frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} f^*(s+a)g^*(s+a+b+1)x^{-s}\,\mathrm{d}s \tag{B.6}$$

which are also referred to as generalized Parseval relations. The quantity on the lhs of equations (B.5) and (B.6) is called the Mellin convolution. In other words, equations (B.5) and (B.6) state that the inverse Mellin transform of a product of two linearly shifted Mellin transforms is equal to the Mellin convolution of individual inverse Mellin transforms.

The Stieltjes moment problem consists in finding a positive $W(x)$ such that for given $\rho(n)$

$$\int_0^\infty x^n W(x)\,\mathrm{d}x = \rho(n) \qquad (n = 0, 1, 2 \ldots \infty). \tag{B.7}$$

A representative application of this problem is the atomic physics where $W(x)$ is the electronic density (see [25] and references therein).

The moment problem can be treated by the inverse Mellin transform [26]. We rewrite equation (B.7) as

$$\int_0^\infty x^{s-1} W(x)\,\mathrm{d}x = \rho(s-1) \qquad s \in \mathbb{C} \tag{B.8}$$

from which

$$W(x) = \mathcal{M}^{-1}[\rho(s-1); x] \tag{B.9}$$

using equation (B.3) for $h = a = 1$ and $b = -1$. If $W(x)$ is positive, it is a solution of the Stieltjes moment problem. The Mellin convolution equation (B.6) allows one to create new soluble moment problems if at least one positive solution of a single moment problem is known.

We illustrate it by choosing in equation (B.6) $a = 0$, $b = -1$ and $f^*(s) = g^*(s)$. If $f(x)$ is a positive function then the inverse Mellin transform of $[f^*(s)]^2$ is equal to $\int_0^\infty f(\frac{x}{t}) f(t) \frac{1}{t}\,\mathrm{d}t$ which is again positive. In other words, having solved equation (B.7) for $\rho_1(n)$ we solved it for $\rho_1^2(n)$, $\rho_1^3(n) \ldots$ etc. As in general the related problems will have non-unique solutions, it follows that a solution of a given problem generates an infinity of solutions of an infinity of related problems.

As an application of the formula equation (B.2), we now present the explicit calculation of the inverse Mellin transform of equation (36),

$$\int_0^\infty x^n W(x)\,\mathrm{d}x = \Gamma(\alpha n + \beta)\sin(\pi k(n+1)) \qquad k = \pm 1, \pm 2 \ldots \tag{B.10}$$

i.e. the calculation of

$$\begin{aligned} W(x) &= \frac{1}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \Gamma(\alpha(s-1)+\beta)\sin(\pi k s) x^{-s}\,\mathrm{d}s \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \Gamma(\alpha(\mathrm{i}y-1)+\beta)\sin(\pi k \mathrm{i}y)\mathrm{e}^{-\mathrm{i}y\ln(x)}\,\mathrm{d}y. \end{aligned} \tag{B.11}$$

In equation (B.11) we use $\Gamma(z) = \int_0^\infty t^{z-1}\mathrm{e}^{-t}\,\mathrm{d}t$ (Euler's representation) and rewrite it as

$$\begin{aligned} W(x) = &\int_0^{+\infty} \left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} t^{\mathrm{i}\alpha y}\frac{\mathrm{e}^{-\pi k y - \mathrm{i}y\ln(x)}}{2\mathrm{i}}\,\mathrm{d}y\right] t^{-\alpha+\beta-1}\mathrm{e}^{-t}\,\mathrm{d}t \\ &- \int_0^{+\infty} \left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} t^{\mathrm{i}\alpha y}\frac{\mathrm{e}^{\pi k y - \mathrm{i}y\ln(x)}}{2\mathrm{i}}\,\mathrm{d}y\right] t^{-\alpha+\beta-1}\mathrm{e}^{-t}\,\mathrm{d}t. \end{aligned} \tag{B.12}$$

The internal integration in equation (B.12) gives Dirac delta functions, leading to

$$\begin{aligned} W(x) = &\frac{1}{2\mathrm{i}}\int_0^{+\infty} \delta(\alpha\ln(t) - (\ln(x) - \mathrm{i}\pi k)) t^{-\alpha+\beta-1}\mathrm{e}^{-t}\,\mathrm{d}t \\ &- \frac{1}{2\mathrm{i}}\int_0^{+\infty} \delta(\alpha\ln(t) - (\ln(x) + \mathrm{i}\pi k)) t^{-\alpha+\beta-1}\mathrm{e}^{-t}\,\mathrm{d}t. \end{aligned}$$

Integrating out the δ leads to

$$\begin{aligned} W(x) = &\frac{1}{2\mathrm{i}\alpha}\int_{-\infty}^{+\infty} [\delta(u - (\ln(x) - \mathrm{i}\pi k)) - \delta(u - (\ln(x) + \mathrm{i}\pi k))] \\ &\times \exp\left(-\mathrm{e}^{\frac{u}{\alpha}} + \frac{u}{\alpha}(\beta - \alpha)\right)\,\mathrm{d}u \end{aligned} \tag{B.13}$$

which after a few steps gives

$$W(x) = \frac{x^{\frac{\beta-\alpha}{\alpha}} \mathrm{e}^{-x^{\frac{1}{\alpha}} \cos\left(\frac{\pi k}{\alpha}\right)}}{\alpha} \sin\left(\pi k\left(1-\frac{\beta}{\alpha}\right) + x^{\frac{1}{\alpha}} \sin\left(\frac{\pi k}{\alpha}\right)\right). \tag{B.14}$$

The $W(x)$ in the above form is integrable only if $\cos\left(\frac{\pi k}{\alpha}\right) > 0$, which requires $\alpha > 2|k|$.

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